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A rational analogue of the Krall polynomials

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Abstract

Krall's polynomials are orthogonal polynomials that are also eigenfunctions of a differential operator. We exhibit an analogue of Krall's polynomials within the context of rank-one commutative rings of difference operators. The corresponding spectral curves are unicursal curves with equations $v^2 = u^{2R+1}(u+1)^{2S+1}$, $R = 0, 1, 2, \ldots,$ $S = 0, 1, 2, \ldots.$ Our analogues of Krall's polynomials are rational functions, which satisfy an orthogonality relation on the circle. The proof of the orthogonality relations combines the discrete Kadomtsev–Petviashvili bilinear identities, the cuspidal character of the singularities of the spectral curves, together with an extra symmetry of the problem.

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1. Introduction

In 1938, Krall [11] posed the problem to determine all families of orthogonal polynomials which are eigenfunctions of a differential operator of an arbitrary order. Already in 1929, Bochner [1] had proved that, when the operator is of order two, the only solutions are provided by the classical orthogonal polynomials: the Hermite, the Laguerre, the Jacobi and the lesser known Bessel polynomials. In general, Krall showed that the operator has to be of even order, and in [12] he solved the problem completely in the case of an operator of order four. In the last few years, there has been a renewal of interest in Krall's problem, see [3–5, 9, 10, 17].

A version of Krall's problem, purely within the context of differential operators, was proposed some time ago by Duistermaat and Grünbaum [2]. This work is now widely known under the name of 'the bispectral problem'. In [16], Wilson proposed to classify bispectral commutative rings of differential operators and he was able to solve the problem completely in the case of rank one. We remind the reader that, by definition, the rank is the greatest common divisor of the orders of all the operators in the ring. In [7], we started a study of bispectral commutative rings of difference operators, for which there is a family of eigenfunctions that are also eigenfunctions of a differential operator in the spectral variable. This work led us

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to conjecture an analogue of Wilson's result [16]. Namely, a rank-one commutative ring of difference operators is bispectral if and only if the spectrum of this ring (in the sense of algebraic geometry) is a *unicursal* curve that compactifies by adding *two* non-singular points at infinity (instead of *one* non-singular point in the purely differential version of the problem). The 'if' part of the conjecture is proved in [7]. Paraphrasing Wilson [16, p 188], the 'only if' part of the conjecture is 'very plausible', but the proof is 'quite troublesome', and it still needs to be worked out in detail, see remark 2.3.

Motivated by Krall's problem, the aim of this paper is to construct *explicitly all* maximal rank-one commutative rings of difference operators whose spectrum is a unicursal curve, and which contain a tridiagonal matrix with one diagonal above and one diagonal below the main diagonal. An easy argument shows that these rings must necessarily correspond to the class of unicursal curves with equations

$$v^2 = u^{2R+1}(u+1)^{2S+1} \qquad R = 0, 1, 2, \ldots \quad S = 0, 1, 2, \ldots . \tag{1.1}$$

This result is established in proposition 2.4. Then, we show that the tridiagonal matrices that are contained in these rings can be obtained explicitly by performing a sequence of Darboux transformations from the discrete second-derivative operator

$$L_0 f_n = f_{n+1} - 2f_n + f_{n-1}$$

adding bound states at *two* different points, with multiplicity R and S, respectively. The precise statement is given in theorem 4.2.

When introducing a rational parameter x with $x = 0$ and ∞ corresponding to the two points at infinity of the curves in (1.1), a suitably normalized common eigenfunction $p_n(x)$ to the operators in the rings above, provides a family of rational functions of x with poles at $0, \infty$ and the two cusps $x = \pm 1$. These rational functions satisfy an orthogonality relation on any simple closed curve encircling the origin and avoiding ± 1, thus providing a rational analogue of Krall's orthogonal polynomials. The proof of the orthogonality relations depends on relating the Darboux process with the notion of the wave and adjoint wave operators of the discrete Kadomtsev–Petviashvili (KP) hierarchy, as well as exploiting the cuspidal character of the singular points of the curves (1.1). These results form the content of theorems 3.3 and 5.2, while section 2 reviews the necessary background material.

The solutions described in this paper are reminiscent of the rank-one solutions of the purely continuous version of the bispectral problem, as formulated in [2]. There, *all* rank-one solutions are obtained by iteration of the Darboux transformation starting from $\mathrm{d}^2/\mathrm{d}x^2$. The spectrum of the corresponding rings are unicursal curves with equations $v^2 = u^{2R+1}$, $R = 0, 1, 2, \ldots$. A proof of the 'only if' part of the conjecture formulated at the beginning of the introduction, would show that we have found *all* bispectral rank-one commutative rings of difference operators that contain a tridiagonal matrix.

In a recent work [14], Nijhoff and Chalykh have conjectured a purely difference analogue of Wilson's result [16] and established it in a generic situation. It would be interesting to understand how the solutions obtained in the present paper relate to their approach, by some limiting procedure.

2. The ΔKP hierarchy and the adelic flag manifold

We start by recalling a few basic facts concerning the discrete KP (ΔKP) hierarchy and the adelic flag manifold, that will play a crucial role in the rest of our paper. The proofs can all be found in our previous work [7] and the pioneering paper [16].

The discrete KP hierarchy is the family of evolution equations in infinitely many time variables $t = (t_1, t_2, t_3, \ldots)$

$$\frac{\partial L}{\partial t_i} = [(L^i)_+, L] \tag{2.1}$$

for L a general first-order formal pseudo-difference operator

$$L = \Delta + \sum_{j=0}^{\infty} a_j(n)\Delta^{-j} = \Lambda + (a_0(n) - 1)I + \sum_{j=1}^{\infty} b_j(n)\Lambda^{-j}$$

with $(L^i)_+$ the positive difference part of L^i. Here Λ and Δ denote, respectively, the customary shift and difference operators acting on the ring $\mathfrak{R}$ of functions of a discrete variable $n \in \mathbb{Z}$ by

$$\Lambda f(n) = f(n+1) \qquad \text{and} \qquad \Delta f(n) = f(n+1) - f(n) = (\Lambda - I)f(n)$$

where I is the identity matrix. In the following, we denote by X^* the adjoint (i.e. the transpose, if we think in terms of matrices) of a formal pseudo-difference operator X.

A Sato-type theory for this hierarchy was developed in [7] (see also [8]), by conjugating L to the difference operator Δ, that is $L = W\Delta W^{-1}$, with

$$W(n; t) = 1 + \sum_{j=1}^{\infty} w_j(n; t)\Delta^{-j}$$

the so-called wave operator. The wavefunction $w(n; t, z)$ and the adjoint wavefunction $w^*(n; t, z)$ are, respectively, defined by

$$w(n; t, z) = W(n; t)\,\mathrm{Exp}(n; t, z) \tag{2.2}$$

$$w^*(n; t, z) = (W^{-1}(n-1; t))^*\,\mathrm{Exp}^{-1}(n; t, z) \tag{2.3}$$

where $\mathrm{Exp}(n; t, z)$ denotes the exponential function

$$\mathrm{Exp}(n; t, z) = (1+z)^n \exp\left(\sum_{i=1}^{\infty} t_i z^i\right). \tag{2.4}$$

A key ingredient of the theory are the so-called bilinear identities.

ΔKP bilinear identities (see proposition 2.1 in [7]).

$$\mathrm{res}_{z=\infty}\, w(n; t, z)w^*(m; t', z)\,\mathrm{d}z = 0 \qquad \forall n \geqslant m \quad \text{and} \quad \forall t, t'. \tag{2.5}$$

These identities imply the existence of a tau function $\tau(n; t)$, such that

$$w(n; t, z) = \frac{\tau(n; t - [z^{-1}])}{\tau(n; t)}\,\mathrm{Exp}(n; t, z) \tag{2.6}$$

and

$$w^*(n; t, z) = \frac{\tau(n; t + [z^{-1}])}{\tau(n; t)}\,\mathrm{Exp}^{-1}(n; t, z) \tag{2.7}$$

where $[z] = (z, z^2/2, z^3/3, \ldots)$.

We denote by $e(r, \lambda)$ the linear functional acting on a function $g(z)$ by the formula

$$\langle e(r, \lambda), g\rangle = g^{(r)}(\lambda) \qquad \lambda \in \mathbb{C} \quad r \geqslant 0. \tag{2.8}$$

Following [16], a linear functional of the type $c = \sum_{r,\text{ finite}} \alpha_r e(r, \lambda)$, will be called a *one-point condition* since, when applied to a function g, it involves the values of this function and its derivatives at only one point $\lambda \in \mathbb{C}$.

We shall denote by Wr_Δ the discrete Wronskian with respect to the variable n, also known as a Casorati determinant:

$$\mathrm{Wr}_\Delta\big(\phi_1(n), \phi_2(n), \ldots, \phi_K(n)\big) = \det\big(\Delta^{i-1}\phi_j(n)\big)_{1\leqslant i,j\leqslant K}. \tag{2.9}$$

Definition 2.1. *Given a family of one-point conditions c_j, $1 \leqslant j \leqslant K$, at the (not necessarily distinct) points $\lambda_1, \ldots, \lambda_K$, we call*

$$\tau(n; t) = \mathrm{Wr}_\Delta(\phi_1(n; t), \ldots, \phi_K(n; t)) \exp\bigg(-\sum_{i=1}^{\infty} t_i \sum_{j=1}^{K} \lambda_j^i\bigg) \prod_{j=1}^{K} (1 + \lambda_j)^{-n} \tag{2.10}$$

with

$$\phi_j(n; t) = \langle c_j, \mathrm{Exp}(n; t, z)\rangle$$

an adelic tau function of the ΔKP hierarchy.

The purpose of the exponential factor in (2.10) is to cancel the exponential factor that comes out automatically from the Wronskian determinant. With this normalization, the tau function $\tau(n; t)$ becomes a (quasi-)polynomial in all variables, n and $\{t_j\}$. The tau function $\tau(n; t)$ can be viewed as an infinite sequence (indexed by n) of tau functions of the standard KP hierarchy, associated with a flag of nested subspaces

$$\mathcal{V}: \quad \cdots \subset V_{n+1} \subset V_n \subset V_{n-1} \subset \cdots \tag{2.11}$$

with

$$V_n = \text{span of } \{w(n; 0, z), w(n+1; 0, z), w(n+2; 0, z), \ldots\}. \tag{2.12}$$

The plane V_0 belongs to the adelic Grassmannian of Wilson [16], that is

$$V_0 = \prod_{j=1}^{K} (z - \lambda_j)^{-1} V_C \qquad \text{with} \quad V_C = \{g \in \mathbb{C}[z] : \langle c_j, g\rangle = 0,\ 1 \leqslant j \leqslant K\} \tag{2.13}$$

where $\mathbb{C}[z]$ denotes the space of polynomials in z. The corresponding flags (2.11) form the so-called *adelic flag manifold*, introduced in our previous work [7].

For a plane V_0 in the adelic Grassmannian as in (2.13), we denote by A_{V_0} the ring of polynomials that leave V_0 invariant, i.e.

$$A_{V_0} = \{p(z) \in \mathbb{C}[z] : p(z)V_0 \subset V_0\}. \tag{2.14}$$

If $\mathcal{V}$ is the associated flag (2.11), we introduce the ring $A_{\mathcal{V}}$ of rational functions that preserve the flag $\mathcal{V}$:

$$A_{\mathcal{V}} = \big\{f(z) \in \mathbb{C}(z) \text{ with poles only at } z = -1 \text{ and } z = \infty: \\ \exists k \in \mathbb{Z} \text{ for which } f(z)V_n \subset V_{n+k}, \forall n\big\}. \tag{2.15}$$

It is shown in [7] that, for each $f \in A_{\mathcal{V}}$, there is a finite band operator L_f with i diagonals above the main diagonal and j diagonals below it, with i and j denoting, respectively, the order of the poles of f at $z = \infty$ and -1, such that $L_f w(n; t, z) = f(z)w(n; t, z)$. The curve $\mathrm{Spec}(A_{\mathcal{V}})$ is a unicursal curve that completes by adding two non-singular points at infinity. Precisely, the coordinate z defines a bijective birational map from $\mathrm{Spec}(A_{\mathcal{V}})$ to $\mathbb{C} \setminus \{-1\}$. The adelic flag manifold parametrizes the maximal rank-one commutative rings of difference operators whose spectrum is a unicursal curve. In the following, we make a slight abuse of the notation by also using A_V and $A_{\mathcal{V}}$ to denote the corresponding rings of differential and difference operators.

We shall denote by $\tilde{w}(n, z)$ the reduced wavefunction

$$\tilde{w}(n, z) = w(n; t, z) \exp\left(-\sum_{i=1}^{\infty} t_i z^i\right). \tag{2.16}$$

Theorem 2.2 (see theorem 5.2 in [7]). *Let $\tau(n; t)$ be an adelic tau function of the ΔKP hierarchy. Then, the function $w'(y, z)$ defined by*

$$w'(y, z) = \tilde{w}(z, \mathrm{e}^y - 1) \tag{2.17}$$

with $\tilde{w}(n, z)$ defined as in (2.6) and (2.16), is the wavefunction of a maximal rank-one commutative ring of differential operators $A_{V'}$ in the variable y, with $\mathrm{Spec}(A_{V'})$ a rational curve. Consequently, the common eigenfunction $\tilde{w}(n, z)$ of the maximal rank-one commutative ring of difference operators L_f, satisfying

$$L_f \tilde{w}(n, z) = f(z)\tilde{w}(n, z) \qquad \forall f \in A_{\mathcal{V}}$$

is also the common eigenfunction of a maximal rank-one commutative ring of differential operators B_θ (in the variable y or $z = \mathrm{e}^y - 1$), that is

$$B_\theta\left(z, \frac{\mathrm{d}}{\mathrm{d}z}\right)\tilde{w}(n, z) = \theta(n)\tilde{w}(n, z) \qquad \forall \theta \in A_{V'}. \tag{2.18}$$

Remark 2.3. As mentioned in the introduction, theorem 2.2 establishes the 'if' part of our conjecture that a maximal rank-one commutative ring of difference operators is bispectral (with a differential ring as dual ring) if and only if the spectrum of this ring is a unicursal curve, which completes by adding two non-singular points at infinity. As in [16], one can show that the coefficients of the operators in a bispectral ring of difference operators, must be rational functions of the discrete variable n. Proving the 'only if' part of the conjecture amounts then to proving that, in the case of a rank-one commutative ring of difference operators, the coefficients of the operators in the ring can be rational functions of n only if the spectrum of the ring is a unicursal curve.

Proposition 2.4. *Let $\mathcal{V}$ be an adelic flag. The corresponding maximal rank-one commutative ring of difference operators contains a tridiagonal matrix (with one diagonal above and one diagonal below the main diagonal) only if* $\mathrm{Spec}(\mathrm{A}_{\mathcal{V}})$ *has an equation as in (1.1).*

Proof. By assumption, there exists a function $u \in A_{\mathcal{V}}$ with a simple pole at $z = \infty$ and -1. Let $w \in A_{\mathcal{V}}$ be another function corresponding to a non-trivial operator commuting with the tridiagonal matrix L_u. Using u (and its powers), we can always kill the pole of w at $z = -1$ and assume that w only has a pole of order $N \geqslant 1$ at $z = \infty$. We can also suppose that N is minimal. The functions $w_0 = w$, $w_1 = (w_0 - w_0(-1))u \equiv wu + c_1 u$, $w_2 = (w_1 - w_1(-1))u \equiv wu^2 + c_1 u^2 + c_2 u, \ldots$, have a pole only at ∞ of order $N, N+1, N+2, \ldots$, respectively. By taking an appropriate linear combination, the function $w^2 + \sum_{i=0}^{N} \alpha_i w_i$ will have a pole of order $N - 1$ at ∞. By the minimality assumption, it must therefore be identically constant, i.e. $w^2 + \sum_{i=0}^{N} \alpha_i w_i = c \Leftrightarrow v^2 = P_{2N}(u)$, with $v = 2w + \sum_{i=0}^{N} \alpha_i u^i$, and $P_{2N}(u)$ some polynomial in u of degree $2N$. A curve of the form $v^2 = P_{2N}(u)$ can be rationally parametrized only if the polynomial $P_{2N}(u)$ has one or two roots of odd multiplicities. If, furthermore, this curve is unicursal, it cannot have roots of even multiplicities. This shows that $\mathrm{Spec}(\mathrm{A}_{\mathcal{V}})$ has an equation of the form (1.1), which concludes the proof. $\square$

3. A factorization problem

Let

$$L_0 = \Lambda - 2I + \Lambda^{-1}. \tag{3.1}$$

The lattice version of the elementary Darboux transformation [13] (see also [3, 4]), amounts to performing a lower–upper factorization of the doubly infinite matrix L_0 and to producing a new matrix by exchanging the order of the factors. The factorization involves a free parameter that is present in the new operator. In this section, we compute explicitly the result of the chain of elementary Darboux transformations

$$\begin{aligned} &L_0 = P_0 Q_0 \mapsto L_1 = Q_0 P_0 = P_1 Q_1 \mapsto \cdots \mapsto L_R = Q_{R-1} P_{R-1} \\ &L_R + 4I = P_R Q_R \mapsto L_{R,1} + 4I = Q_R P_R \\ &\qquad = P_{R+1} Q_{R+1} \mapsto \cdots \mapsto L_{R,S} + 4I = Q_{R+S-1} P_{R+S-1}. \end{aligned} \tag{3.2}$$

It is equivalent to performing a lower–upper factorization of the operator

$$L_0^R (L_0 + 4I)^S = (\Lambda^{-(R+S)} P) Q \tag{3.3}$$

where P and Q denote monic positive difference operators of order $R + S$, with the kernel of Q specified by $R + S$ functions $\phi_1, \phi_2, \ldots, \phi_R$ and $\psi_1, \psi_2, \ldots, \psi_S$ satisfying

$$L_0 \phi_j = \phi_{j-1} \qquad 1 \leqslant j \leqslant R \tag{3.4}$$

$$(L_0 + 4I)\psi_j = \psi_{j-1} \qquad 1 \leqslant j \leqslant S \tag{3.5}$$

with the convention that $\phi_0 = 0$ and $\psi_0 = 0$. Indeed, from (3.4) and (3.5), we have that $\operatorname{Ker} Q \subset \operatorname{Ker} QL_0$, implying that QL_0 can be factorized to the right by Q, i.e.

$$QL_0 = L_{R,S} Q \tag{3.6}$$

where $L_{R,S}$ must necessarily be a tridiagonal operator. It is easy to check that this operator coincides with the one resulting from the sequence of elementary Darboux transformations (3.2) (see section 3.3 of [18], where a similar argument is used).

In order to solve the factorization problem (3.3), we shall need the functions

$$S_j^\varepsilon(n; t) = \frac{1}{j!} \langle e(j, \varepsilon - 1), \operatorname{Exp}(n; t, z) \rangle \tag{3.7}$$

with $\operatorname{Exp}(n; t, z)$ and $e(j, \lambda)$ defined as in (2.4) and (2.8), respectively. When $\varepsilon = 1$, the functions $S_j^1(n; t)$ are a shifted version of the classical elementary Schur polynomials, defined by $\exp\left(\sum_{i=1}^\infty t_i z^i\right) = \sum_{j=0}^\infty S_j(t) z^j$:

$$S_j^1(n; t) = S_j(t_1 + n, t_2 - n/2, t_3 + n/3, \ldots).$$

A q-version of these polynomials was first considered by us in [6], with the discrete derivative replaced by a q-derivative. It will be convenient to use the notation

$$L_0^\varepsilon = \Lambda - 2\varepsilon I + \varepsilon^2 \Lambda^{-1}. \tag{3.8}$$

Lemma 3.1. *Let us define*

$$\phi_j^\varepsilon(n; t) = S_{2j-1}^\varepsilon(n + j - 1; t). \tag{3.9}$$

Then

$$L_0^\varepsilon \phi_j^\varepsilon = \phi_{j-1}^\varepsilon. \tag{3.10}$$

Proof. From the definition (3.7), we have

$$
\begin{aligned}
S_j^{\varepsilon}(n+1;t) - \varepsilon S_j^{\varepsilon}(n;t) &= \frac{1}{j!}\langle e(j,\varepsilon-1), (z+1-\varepsilon)\,\mathrm{Exp}(n;t,z)\rangle \\
&= \frac{1}{(j-1)!}\langle e(j-1,\varepsilon-1), \mathrm{Exp}(n;t,z)\rangle \\
&= S_{j-1}^{\varepsilon}(n;t).
\end{aligned} \tag{3.11}
$$

By repeated use of this identity, we find

$$
\begin{aligned}
L_0^{\varepsilon}\phi_j^{\varepsilon}(n;t) &= S_{2j-1}^{\varepsilon}(n+j;t) - \varepsilon S_{2j-1}^{\varepsilon}(n+j-1;t) \\
&\quad -\varepsilon(S_{2j-1}^{\varepsilon}(n+j-1;t) - \varepsilon S_{2j-1}^{\varepsilon}(n+j-2;t)) \\
&= S_{2j-2}^{\varepsilon}(n+j-1;t) - \varepsilon S_{2j-2}^{\varepsilon}(n+j-2;t) \\
&= S_{2j-3}^{\varepsilon}(n+j-2;t) \\
&= \phi_{j-1}^{\varepsilon}(n;t)
\end{aligned}
$$

which establishes the lemma. □

We denote by

$$
\phi_j(n;t) = \phi_j^{1}(n;t) \qquad \text{and} \qquad \psi_j(n;t) = \phi_j^{-1}(n;t) \tag{3.12}
$$

the functions $\phi_j^{\varepsilon}(n;t)$ with $\varepsilon = 1$ and -1, as defined in (3.9). From (3.4), (3.5) and the previous lemma, it follows that the operator Q in (3.3) is given explicitly by

$$
Q(n;t)f(n) = \frac{\mathrm{Wr}_{\Delta}(\phi_1(n;t),\ldots,\phi_R(n;t),\psi_1(n;t),\ldots,\psi_S(n;t),f(n))}{\mathrm{Wr}_{\Delta}(\phi_1(n;t),\ldots,\phi_R(n;t),\psi_1(n;t),\ldots,\psi_S(n;t))} \tag{3.13}
$$

where Wr_{Δ} denotes the discrete Wronskian defined in (2.9). We define

$$
\begin{aligned}
\tau(n;t) &= \mathrm{Wr}_{\Delta}(\phi_1(n;t),\ldots,\phi_R(n;t),\psi_1(n;t),\ldots,\psi_S(n;t)) \\
&\quad \times(-1)^{nS}\exp\left(-S\sum_{i=1}^{\infty} t_i(-2)^i\right).
\end{aligned} \tag{3.14}
$$

Rewriting (3.11) as

$$
S_j^{\varepsilon}(n+1;t) = \varepsilon S_j^{\varepsilon}(n;t) + S_{j-1}^{\varepsilon}(n;t)
$$

and remembering the definition (3.9), we deduce easily that

$$
\phi_j^{\varepsilon}(n;t) = \sum_{i=0}^{j-1}\binom{j-1}{i}\varepsilon^{j-1-i}S_{2j-1-i}^{\varepsilon}(n;t) \equiv \langle c_j^{\varepsilon}, \mathrm{Exp}(n;t,z)\rangle \tag{3.15}
$$

with the linear functionals c_j^{ε} (acting on functions of z) defined by

$$
\langle c_j^{\varepsilon}, g(z)\rangle = \sum_{i=0}^{j-1}\binom{j-1}{i}\frac{\varepsilon^{j-i-1}}{(2j-i-1)!}e(2j-i-1,\varepsilon-1). \tag{3.16}
$$

Combining (3.12), (3.15) and (3.16), we have shown that $\tau(n;t)$ in (3.14) is an adelic tau function (in the sense of definition 2.1) of the ΔKP hierarchy defined by $R+S$ one-point conditions, with R conditions at the point zero and S conditions at the point -2. Namely, V_C in (2.13) is given by

$$
V_C = \left\{g \in \mathbb{C}[z] : \langle c_j^1, g(z)\rangle = 0,\ 1 \leqslant j \leqslant R \text{ and } \langle c_j^{-1}, g(z)\rangle = 0,\ 1 \leqslant j \leqslant S\right\}. \tag{3.17}
$$

In order to obtain an explicit formula for the factor P in (3.3), we shall need the following technical lemma.

Lemma 3.2. *Let* $\varepsilon = \pm 1$. *Then*

$$\phi_j^\varepsilon(n;t) - \frac{1}{z}\Delta^*\phi_j^\varepsilon(n;t) = \left(\frac{z+1-\varepsilon}{z}\right)^2 \phi_j^\varepsilon(n;t+[z^{-1}]) - \frac{z+1}{z^2}\phi_{j-1}^\varepsilon(n;t+[z^{-1}]). \quad (3.18)$$

Proof. From (3.8) and (3.10), we have that

$$\phi_j^\varepsilon(n+1;t) + \varepsilon^2\phi_j^\varepsilon(n-1;t) = 2\varepsilon\phi_j^\varepsilon(n;t) + \phi_{j-1}^\varepsilon(n;t). \quad (3.19)$$

Using (3.15), we also have that

$$\phi_j^\varepsilon(n;t-[z^{-1}]) = \frac{z+1}{z}\phi_j^\varepsilon(n;t) - \frac{1}{z}\phi_j^\varepsilon(n+1;t). \quad (3.20)$$

When $\varepsilon = \pm 1$, combining (3.20) and (3.19) (in that order), we deduce that

$$\begin{aligned}
\phi_j^\varepsilon(n;t-[z^{-1}]) - \frac{1}{z}\Delta^*\phi_j^\varepsilon(n;t-[z^{-1}]) &= \frac{z+1}{z}\phi_j^\varepsilon(n;t-[z^{-1}]) - \frac{1}{z}\phi_j^\varepsilon(n-1;t-[z^{-1}]) \\
&= \frac{z+1}{z}\left(\frac{z+1}{z}\phi_j^\varepsilon(n;t) - \frac{1}{z}\phi_j^\varepsilon(n+1;t)\right) - \frac{1}{z}\left(\frac{z+1}{z}\phi_j^\varepsilon(n-1;t) - \frac{1}{z}\phi_j^\varepsilon(n;t)\right) \\
&= \frac{(z+1)^2+1}{z^2}\phi_j^\varepsilon(n;t) - \frac{z+1}{z^2}\left(\phi_j^\varepsilon(n+1;t) + \phi_j^\varepsilon(n-1;t)\right) \\
&= \frac{(z+1)^2+1}{z^2}\phi_j^\varepsilon(n;t) - \frac{z+1}{z^2}\left(2\varepsilon\phi_j^\varepsilon(n;t) + \phi_{j-1}^\varepsilon(n;t)\right) \\
&= \left(\frac{z+1-\varepsilon}{z}\right)^2\phi_j^\varepsilon(n;t) - \frac{z+1}{z^2}\phi_{j-1}^\varepsilon(n;t).
\end{aligned}$$

Substituting $t+[z^{-1}]$ for t in this last formula gives (3.18), concluding the proof. □

In the next theorem, we give an explicit formula for the operator P appearing in (3.3), which together with (3.13) provides the explicit solution of the factorization problem that we posed at the beginning of the section. Besides, we show that Q and P are intimately related, respectively, with the wave operator and the adjoint wave operator of the ΔKP hierarchy defined by the one-point conditions in (3.17).

Theorem 3.3. *The wave operator and the adjoint wave operator of the* Δ*KP hierarchy corresponding to the* $R+S$ *one-point conditions in (3.17), are expressed in terms of the operators* Q *and* P *solving the factorization problem in (3.3), via the following formulae:*

$$W = Q(\Lambda - I)^{-R}(\Lambda + I)^{-S} \quad (3.21)$$

$$(W^{-1})^* = P^*(\Lambda^* - I)^{-R}(\Lambda^* + I)^{-S}. \quad (3.22)$$

Moreover, we have an explicit Wronskian formula for the adjoint operator P^* *in terms of the functions* $\phi_j(n;t)$ *and* $\psi_j(n;t)$ *defining the operator* Q *in (3.13), namely*

$$P^*(n;t)f(n) = \frac{\mathrm{Wr}_{\Delta^*}(\phi_1^*(n;t),\ldots,\phi_R^*(n;t),\psi_1^*(n;t),\ldots,\psi_S^*(n;t),f(n))}{\mathrm{Wr}_{\Delta^*}(\phi_1^*(n;t),\ldots,\phi_R^*(n;t),\psi_1^*(n;t),\ldots,\psi_S^*(n;t))} \quad (3.23)$$

with

$$\begin{aligned}
\phi_j^*(n;t) &= \phi_j(n+R+S;t) \qquad 1 \leqslant j \leqslant R \\
\psi_j^*(n;t) &= \psi_j(n+R+S;t) \qquad 1 \leqslant j \leqslant S
\end{aligned} \quad (3.24)$$

and Wr_{Δ^*} *defined as in (2.9) with* Δ *replaced by* Δ^*.

Proof. From (3.21) we have that

$$W(n;t)\,\mathrm{Exp}(n;t,z) = z^{-R}(z+2)^{-S}Q\,\mathrm{Exp}(n;t,z). \tag{3.25}$$

Using the definition of Q in (3.13), formula (3.14) for $\tau(n;t)$ and the identity (3.20), by some elementary row manipulations on the discrete Wronskian determinant which appears in the numerator, we find that (3.25) agrees with (2.6).

Substituting (3.21) into (3.3), we find that

$$L_0^R(L_0+4I)^S = \Lambda^{-(R+S)}PW(\Lambda-I)^R(\Lambda+I)^S.$$

Since

$$L_0^R(L_0+4I)^S = \Lambda^{-(R+S)}(\Lambda-I)^{2R}(\Lambda+I)^{2S}$$

we deduce that

$$P = (\Lambda-I)^R(\Lambda+I)^S W^{-1}.$$

Taking the adjoint of this equation, gives (3.22).

It remains to establish (3.23) and (3.24). It is enough to show that $w^*(n;t,z)$ defined via (2.3), (3.22)–(3.24), satisfies (2.7), that is, we have to prove that

$$P^*(n-1;t)\,\mathrm{Exp}^{-1}(n;t,z) = z^R(z+2)^S\frac{\tau(n;t+[z^{-1}])}{\tau(n;t)}\,\mathrm{Exp}^{-1}(n;t,z) \tag{3.26}$$

with $\tau(n;t)$ as in (3.14). One checks easily that

$$\begin{aligned}\tau(n;t) = (-1)^{nS+(R+S)(R+S-1)/2}\exp\left(-S\sum_{i=1}^{\infty}t_i(-2)^i\right)\\ \times\mathrm{Wr}_{\Delta^*}(\phi_1^*(n-1;t),\ldots,\phi_R^*(n-1;t),\psi_1^*(n-1;t),\ldots,\psi_S^*(n-1;t)).\end{aligned} \tag{3.27}$$

We replace row i by row $i-(1/z)\times$ row $(i+1)$, for $1\leqslant i\leqslant R+S$, in the discrete Wronskian determinant on the numerator of the left-hand side of (3.26), using (3.18) at each step (with $\varepsilon=\pm1$); then we expand the determinant along the last column, all of whose entries are zero, except the final one which is $z^{R+S}\,\mathrm{Exp}^{-1}(n;t,z)$. Using (3.27), this gives

$$\begin{aligned}\text{left-hand side of (3.26)} = \frac{(-1)^{nS+(R+S)(R+S-1)/2}\exp\left(-S\sum_{i=1}^{\infty}t_i(-2)^i\right)}{\tau(n;t)}z^{R+S}\\ \times\mathrm{Exp}^{-1}(n;t,z)\,\mathrm{Wr}_{\Delta^*}\Bigg(\phi_1^*(n-1;t+[z^{-1}]),\phi_2^*(n-1;t+[z^{-1}])\\ -\frac{z+1}{z^2}\phi_1^*(n-1;t+[z^{-1}]),\ldots,\phi_R^*(n-1;t+[z^{-1}])\\ -\frac{z+1}{z^2}\phi_{R-1}^*(n-1;t+[z^{-1}]),\left(\frac{z+2}{z}\right)^2\psi_1^*(n-1;t+[z^{-1}]),\\ \left(\frac{z+2}{z}\right)^2\psi_2^*(n-1;t+[z^{-1}])-\frac{z+1}{z^2}\psi_1^*(n-1;t+[z^{-1}]),\ldots,\\ \left(\frac{z+2}{z}\right)^2\psi_S^*(n-1;t+[z^{-1}])-\frac{z+1}{z^2}\psi_{S-1}^*(n-1;t+[z^{-1}])\Bigg).\end{aligned}$$

If we now replace in the right-hand side of the above equation column j by column $j+(z+1)/z^2\times$ column $(j-1)$, for $2\leqslant j\leqslant R$, and column j by column $j+(z+1)/(z+2)^2\times$ column $(j-1)$, for $R+2\leqslant j\leqslant R+S$, we obtain that

$$\text{the left-hand side of (3.26)} = \frac{(-1)^{nS+(R+S)(R+S-1)/2}\exp\left(-S\sum_{i=1}^{\infty}t_i(-2)^i\right)}{\tau(n;t)}z^{R+S}\left(\frac{z+2}{z}\right)^{2S}$$
$$\times\,\mathrm{Exp}^{-1}(n;t,z)\,\mathrm{Wr}_{\Delta^*}\big(\phi_1^*(n-1;t+[z^{-1}]),\ldots,\phi_R^*(n-1,t+[z^{-1}]),$$
$$\psi_1^*(n-1;t+[z^{-1}]),\ldots,\psi_S^*(n-1;t+[z^{-1}])\big).$$

Combining this last equation with (3.27) gives (3.26), establishing the theorem. □

4. The spectral curve

It follows from theorem 3.3 that the eigenfunction of $L_{R,S}$ in (3.2)

$$p_n(x)=W(n;t)x^n=(x-1)^{-R}(x+1)^{-S}Q(n;t)x^n \tag{4.1}$$

with eigenvalue $(x-1)^2/x$ $(x=z+1)$, can be interpreted as a reduced wavefunction, as defined in (2.16), of the ΔKP hierarchy, built from an adelic tau function in the sense of definition 2.1, corresponding to the $R+S$ one-point conditions (3.17). Thus, by theorem 2.2, the tridiagonal matrix $L_{R,S}$ belongs to a maximal rank-one bispectral commutative ring of difference operators.

In this section, we construct explicitly the maximal (rank-one) ring of operators which commute with $L_{R,S}$, and we show that its spectrum is given by (1.1).

Lemma 4.1. *Let $\mathcal{V}$ be the flag defined by the adelic tau function $\tau(n;t)$ in (3.14) corresponding to the $R+S$ one-point conditions in (3.17). Then, the functions*

$$\varphi_j(z)=(z+1)^{-j}z^{2R}(z+2)^{2S}\qquad j\in\mathbb{Z} \tag{4.2}$$

belong to the ring $A_{\mathcal{V}}$ defined in (2.15).

Proof. Let us consider the operator

$$\mathcal{L}=W\mathcal{L}_0W^{-1}\qquad\text{with}\quad \mathcal{L}_0=(\Lambda-I)^{2R}(\Lambda+I)^{2S}\Lambda^{-j}\quad j\in\mathbb{Z} \tag{4.3}$$

and W the wave operator in (3.21). Using (3.21) and (3.22), since constant coefficient pseudo-difference operators commute between themselves, we obtain that $\mathcal{L}=Q\Lambda^{-j}P$. Since P and Q are positive difference operators, this shows that $\mathcal{L}$ in (4.3) is a *finite band* difference operator. Clearly,

$$\mathcal{L}w(n;t,z)=(z+1)^{-j}z^{2R}(z+2)^{2S}w(n;t,z)$$

with $w(n;t,z)$ as in (2.2). Remembering the definition of the flag $\mathcal{V}$ in (2.11) and (2.12), this shows that the functions $\varphi_j(z)$ in (4.2) belong to the ring $A_{\mathcal{V}}$ as defined in (2.15). This establishes the lemma. □

Theorem 4.2. *The tridiagonal operator $L_{R,S}$ constructed in section 3 belongs to a maximal rank-one commutative ring of difference operators $A_{\mathcal{V}}$ with the spectral curve*

$$\mathrm{Spec}(A_{\mathcal{V}}):\qquad v^2=u^{2R+1}(u+1)^{2S+1}. \tag{4.4}$$

Proof. From (3.6) and (3.21), we deduce that $L_{R,S} = WL_0W^{-1}$, with L_0 as in (3.1), which shows that the function $f = z^2/(z+1)$ belongs to $A_{\mathcal{V}}$. This function has a simple pole at $z = -1$ and ∞. Thus, we can assume that all other generators of $A_{\mathcal{V}}$ can be taken to be polynomials, that is they must belong to the ring A_{V_0} (2.14), with $V_0 = z^{-R}(z+2)^{-S}V_C$ the plane in (2.13) defined by the $R+S$ one-point conditions (3.17). From lemma 4.1, the functions $\varphi_j(z) = (z+1)^{-j}z^{2R}(z+2)^{2S} \in A_{\mathcal{V}}$ for all $j \in \mathbb{Z}$. Using the function f above one can kill the pole of these functions at $z = -1$, and in this way produce polynomials $q_k \in A_{V_0}$ of degree k, for all $k \geqslant R+S+1$. Since the conditions (3.16) always contain derivatives, it is obvious that $1 \in V_C$ and therefore $A_{V_0} \subset V_C$. By definition of the adelic Grassmannian [16], the codimension of V_C in $\mathbb{C}[z]$ is equal to $R+S$. Hence, $V_C = \{1, q_{R+S+1}(z), q_{R+S+2}(z), \ldots\}$ and the algebra A_{V_0} is generated by finitely many of the polynomials q_k, $k \geqslant R+S+1$. We now show that the functions

$$u = \frac{z^2}{4(z+1)} \qquad \text{and} \qquad v = \frac{z^{2R+1}(z+2)^{2S+1}}{4^{R+S+1}(z+1)^{R+S+1}}$$

are enough to generate the algebra $A_{\mathcal{V}}$, which will establish (4.4). Let

$$\tilde{\varphi}_j = \frac{z^{2R+1}(z+2)^{2S}}{(z+1)^j}.$$

We have that

$$\varphi_{R+S} = 4^{R+S}u^R(u+1)^S \qquad \text{and} \qquad \tilde{\varphi}_{R+S} = 2^{2R+2S+1}\left(u^{R+1}(u+1)^S + v\right)$$

and thus $\varphi_{R+S}, \tilde{\varphi}_{R+S} \in \mathbb{C}[u, v]$. Using the relations

$$\varphi_{j-1} = \varphi_j + \tilde{\varphi}_j \qquad \text{and} \qquad \tilde{\varphi}_{j-1} = 4\varphi_{j-1}u + \tilde{\varphi}_j$$

one shows inductively that $\varphi_j, \tilde{\varphi}_j \in \mathbb{C}[u, v]$ for $j \in \mathbb{Z}$, which concludes the proof. □

5. Orthogonality relations

In this section, we derive an orthogonality relation on the circle for the functions $p_n(x)$, as defined in (4.1). The differential $w(n; t, z)w^*(m; t, z)\,\mathrm{d}z$ (that appears in the bilinear identities (2.5)) always extends to a regular differential on the affine curve $\mathrm{Spec}(A_{\mathcal{V}})$, see [7]. The next lemma expresses the fact that the residue of a regular differential at a cusp is always zero (see [15, chapter 4]).

Lemma 5.1. *Let $w(n; t, z)$ and $w^*(n; t, z)$ denote, respectively, the wave and the adjoint wavefunctions built from an adelic tau function, via (2.6) and (2.7). Then, for any point λ_j belonging to the support of the corresponding one-point conditions, we have*

$$\mathrm{res}_{z=\lambda_j}\, w(n; t, z)w^*(m; t, z)\,\mathrm{d}z = 0 \qquad \forall n, m \in \mathbb{Z}. \tag{5.1}$$

Theorem 5.2. *The functions $p_n(x)$ defined in (4.1), satisfy the orthogonality relations*

$$\frac{1}{2\pi\mathrm{i}}\oint p_n(x)p_m(x^{-1})\frac{\mathrm{d}x}{x} = \frac{\tau(n+1; t)}{\tau(n; t)}\delta_{nm} \qquad \forall n, m \in \mathbb{Z} \tag{5.2}$$

where the integral can be taken along any simple closed curve surrounding the origin $x = 0$ and avoiding the points $x = \pm 1$, and $\tau(n; t)$ is the tau function defined in (3.14).

Proof. We define the reduced adjoint wavefunction by

$$p_n^*(x) = w^*(n; t, x-1) \exp\left(\sum_{i=1}^{\infty} t_i (x-1)^i\right). \tag{5.3}$$

We first establish the following relation:

$$p_n(x^{-1}) = \frac{\tau(n+1; t)}{\tau(n; t)} x p_{n+1}^*(x) \tag{5.4}$$

which converts (5.2) to

$$\frac{1}{2\pi \mathrm{i}} \oint p_n(x) p_{m+1}^*(x)\, \mathrm{d}x = \delta_{nm} \qquad \forall n, m \in \mathbb{Z}. \tag{5.5}$$

We denote by $(a_{ij}; b_{ij}; x^{c_i})$, $1 \leqslant i \leqslant R+S+1$, the $(R+S+1) \times (R+S+1)$ matrix with entries a_{ij} for the first R columns, entries b_{ij} for the next S columns, and entries x^{c_i} in the $(R+S+1)$th column. With a similar meaning for the $(R+S) \times (R+S)$ matrix $(a_{ij}; b_{ij})$, from (2.2), (2.3), (3.13), (3.21)–(3.23) we compute that

$$p_n(x) = \frac{x^n}{(x-1)^R (x+1)^S} \frac{\det\left(\phi_j(n+i-1; t); \psi_j(n+i-1; t); x^{i-1}\right)}{\det(\phi_j(n+i-1; t); \psi_j(n+i-1; t))} \tag{5.6}$$

$$p_n^*(x) = \frac{(-1)^S x^{-n}}{(1-x)^R (x+1)^S} \frac{\det\left(\phi_j(n+i-2; t); \psi_j(n+i-2; t); x^{R+S+1-i}\right)}{\det(\phi_j(n+i-1; t); \psi_j(n+i-1; t))} \tag{5.7}$$

from which, remembering the definition of $\tau(n; t)$ in (3.14), equation (5.4) follows immediately.

From (5.6) and (5.7), $p_n(x)$ and $p_n^*(x)$ are rational functions of x on the Riemann sphere with poles only at $x = 0, \pm 1$ and ∞. Since the support of the one-point conditions defining the tau function in (3.14) reduces to the points $z = 0$ and -2, i.e. $x = \pm 1$ (since $x = z + 1$), from (5.1), we immediately obtain that $\mathrm{res}_{x=\pm 1}\, p_n(x) p_m^*(x)\, \mathrm{d}x = 0$, $\forall n, m \in \mathbb{Z}$. Thus, in order to establish (5.5), it remains to show that

$$\mathrm{res}_{x=0}\, p_n(x) p_{m+1}^*(x)\, \mathrm{d}x = \delta_{nm} \qquad \forall n, m \in \mathbb{Z}. \tag{5.8}$$

From (5.6) and (5.7), remembering the definition of $\tau(n; t)$ in (3.14), we obtain by a straightforward computation that, around $x = 0$, we have the expansion

$$p_n(x) p_{m+1}^*(x) = \frac{x^{n-m-1}}{\tau(n; t)\tau(m+1; t)} \left(\tau(n+1; t)\tau(m; t) + \mathrm{O}(x)\right)$$

which establishes (5.8) for $m \leqslant n$. The ΔKP bilinear identities (2.5) with $t' = t$, tell us that $\mathrm{res}_{x=\infty}\, p_m(x) p_n^*(x)\, \mathrm{d}x = 0$, $\forall m \geqslant n$. Making the change of variable $y = 1/x$, from (5.4), we obtain that

$$p_n(x) p_{m+1}^*(x)\, \mathrm{d}x = -\frac{\tau(n+1; t)\tau(m; t)}{\tau(n; t)\tau(m+1; t)} p_m(y) p_{n+1}^*(y)\, \mathrm{d}y$$

thus implying (5.8) for $m \geqslant n+1$. This completes the proof of theorem 5.2. □

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